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q-NATURAL TRANSFORMS OF CERTAIN q-HUMBERT FUNCTIONS

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Abstract: The **N**-transform (so-called natural transform) of a function, which may combine Laplace and Sumudu transforms, has been introduced and investigated. Recently q-analogues of the **N**-transform has been presented and studied. Among q-extensions of a number of polynomials and functions, the q-Humbert functions and some of their interesting identities and properties have recently been introduced and provided. In this paper, we aim to present q-natural transforms of a finite product of the q-Humbert functions. Some particular cases of our main identities are also considered.

Keywords and Phrases: Laplace transform, Sumudu transform, Natural transform, q-Laplace transforms, q-Sumudu transforms, q-Natural transforms, q-Gamma functions, q-Humbert functions.

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05A30.

1. Introduction and Preliminaries

Some notations and known facts about q-analogues are recalled (see, e.g., [14], [16], [19], [20], [26, Chapter 6]). The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n=0), \\ \prod_{k=0}^{n-1}\left(1-a\,q^k\right) & (n\in\mathbb{N}), \end{cases} \tag{1.1}$$

where $a,\ q\in\mathbb{C}$ and it is assumed that $a\neq q^{-m}$ $(m\in\mathbb{N}_0)$. Here and in the following, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, positive real numbers, and complex numbers, respectively. Also let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. The q-shifted factorial for non-positive integer subscript is defined by

$$(a;q)_{-n} := \frac{1}{(1-aq^{-1})\,(1-aq^{-2})\cdots(1-aq^{-n})} \quad (n\in\mathbb{N}_0)\,, \tag{1.2}$$

which gives

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n\,q^{\binom{n}{2}}}{(q/a;q)_n} \quad (n\in\mathbb{Z})\,. \tag{1.3}$$

We also recall

$$(a;q)_\infty := \prod_{k=0}^{\infty}\left(1-a\,q^k\right) \quad (a,\ q\in\mathbb{C},\ |q|<1)\,. \tag{1.4}$$

It follows from (1.1), (1.2) and (1.4) that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (n\in\mathbb{Z}), \tag{1.5}$$

which can be extended to $n=\alpha\in\mathbb{C}$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad (\alpha\in\mathbb{C};\ |q|<1). \tag{1.6}$$

Here and elsewhere, the principal value of q^α is assumed.

The q-analogue (or q-extension) of $n\in\mathbb{N}$ is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1} \quad (n\in\mathbb{N}), \tag{1.7}$$

whose extension is given by

$$[z]_q = \frac{1-q^z}{1-q} \quad (z \in \mathbb{C},\, q \in \mathbb{C} \setminus \{1\},\, q^z \neq 1). \tag{1.8}$$

The q-analogue of $n!$ is defined by

$$[n]_q! := \begin{cases} 1 & (n=0), \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.9}$$

The q-binomial coefficient (or the Gaussian polynomial analogous to $\binom{n}{k}$) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n,\, k \in \mathbb{N}_0,\, 0 \leq k \leq n)\,. \tag{1.10}$$

A q-analogue of the classical exponential function e^z is defined by

$$\mathbf{e}_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[(1-q)z]^n}{(q;q)_n} = \frac{1}{((1-q)z;q)_\infty} \quad \left(|z| < \frac{1}{1-q} = [\infty]_q \right), \tag{1.11}$$

and another q-analogue of the classical exponential function e^z is defined by

$$\mathbf{E}_q(z) := \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{[(1-q)z]^n}{(q;q)_n} = (-(1-q)z;q)_\infty \quad (|z| < \infty). \tag{1.12}$$

The q-exponential functions are related as follows:

$$\mathbf{e}_q(-z)\mathbf{E}_q(z) = \mathbf{e}_q(z)\mathbf{E}_q(-z) = 1, \tag{1.13}$$

and

$$\mathbf{e}_{1/q}(z) = \mathbf{E}_q(z). \tag{1.14}$$

Remark 1.1. *The q-exponential functions are often defined by using the following Euler's formulae (see, e.g.,* [1, 2, 5], [26, p. 487]*):*

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \quad (|q| < 1,\ |z| < 1) \tag{1.15}$$

and

$$E_q(z) := \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\binom{n}{2}}\, z^n}{(q;q)_n} = (z;q)_\infty \quad (|q| < 1,\ z \in \mathbb{C})\,. \tag{1.16}$$

We find from (1.11) and (1.12) that

$$\mathbf{e}_q(z) = e_q((1-q)z), \quad e_q(z) = \mathbf{e}_q\left(\frac{z}{1-q}\right) \tag{1.17}$$

and

$$\mathbf{E}_q(z) = E_q(-(1-q)z), \quad E_q(z) = \mathbf{E}_q\left(-\frac{z}{1-q}\right). \tag{1.18}$$

F. H. Jackson [19] may be recognized as the first to develop q-calculus in a systematic way. The q-derivative of a function $f(t)$ is defined by

$$D_q\{f(t)\} := \frac{d_q f(t)}{d_q t} = \frac{f(qt)-f(t)}{(q-1)t}. \tag{1.19}$$

Obviously

$$\lim_{q\to 1} D_q\{f(t)\} = \frac{d}{dt}\{f(t)\},$$

if $f(t)$ is differentiable. Some formulas involving the q-derivative are recalled:

$$D_q\{f(t)g(t)\} = f(qt)\, D_q\{g(t)\} + g(t)\, D_q\{f(t)\}, \tag{1.20}$$

which, upon exchanging f and g, gives

$$D_q\{f(t)g(t)\} = f(t)\, D_q\{g(t)\} + g(qt)\, D_q\{f(t)\}; \tag{1.21}$$

$$D_q\left\{\frac{f(t)}{g(t)}\right\} = \frac{g(qt)D_q\{f(t)\} - f(t)\, D_q\{g(t)\}}{g(t)g(qt)}; \tag{1.22}$$

$$D_q\{t^\alpha\} = [\alpha]_q\, t^{\alpha-1} \quad (\alpha \in \mathbb{C});$$

The following q-analogue of the function $(t+a)^n$ $(n \in \mathbb{N}_0)$ (see [25])

$$(t+a)_q^n = \prod_{j=0}^{n-1}\left(t+q^j a\right) = t^n\left(-\frac{a}{t};q\right)_n, \tag{1.23}$$

an empty product, here and in the following, being conventionally understood to be 1, is the unique solution of the differential equation

$$D_q\left\{(t+a)_q^n\right\} = [n]_q\,(t+a)_q^{n-1}, \quad (t+a)_q^0 = 1; \tag{1.24}$$

Similarly as in (1.6),

$$(1+t)_q^{\alpha} := \frac{(1+t)_q^{\infty}}{(1+q^{\alpha}t)_q^{\infty}} = \frac{(-t;q)_{\infty}}{(-q^{\alpha}t;q)_{\infty}} \quad (t,\alpha \in \mathbb{C};\ |q|<1); \tag{1.25}$$

The following generalizations of (1.24) where $n \in \mathbb{Z}$, and a, b, $\beta \in \mathbb{C}$ are given

$$D_q\left\{(at+b)_q^n\right\} = a\,[n]_q\,(at+b)_q^{n-1}, \tag{1.26}$$

$$D_q\left\{(a+bt)_q^n\right\} = b\,[n]_q\,(a+bqt)_q^{n-1}, \tag{1.27}$$

$$D_q\left\{(1+bt)_q^{\beta}\right\} = b\,[\beta]_q\,(1+bqt)_q^{\beta-1}; \tag{1.28}$$

The chain rule for the usual derivatives is among most useful and important formulas. Regrettably it is noted (see [20, pp. 3–4]) that there does not exist a general chain rule for q-derivatives. Yet, there is an exception taken by

$$D_q\left\{f(u(t))\right\} = D_{q^{\beta}}\left\{f(u)\right\} \cdot D_q\left\{u(t)\right\}, \tag{1.29}$$

where $u = u(t) = \alpha\, t^{\beta}$, α and β being constants.

Suppose that $0 < a < b$. The (Jackson's) definite q-integral is defined as follows (see, e.g., [19], [20, Section 19], [26, Chapter 6]):

$$\int_0^b f(t)\,d_qt = (1-q)\sum_{j=0}^{\infty} q^j b\, f\left(q^j b\right) \tag{1.30}$$

and

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt. \tag{1.31}$$

A more general version of (1.30) is given by

$$\int_0^b f(t)\,d_qg(t) = \sum_{j=0}^{\infty} f\left(q^j b\right)\left(g\left(q^j b\right) - g\left(q^{j+1}b\right)\right). \tag{1.32}$$

The improper q-integral of $f(t)$ on $[0,\infty)$ is defined by

$$\int_0^{\infty} f(t)\,d_qt = (1-q)\sum_{j=-\infty}^{\infty} f\left(q^j\right) q^j \quad (0<q<1) \tag{1.33}$$

and

$$\int_0^{\infty} f(t)\, d_q t = \frac{q-1}{q} \sum_{j=-\infty}^{\infty} f\left(q^j\right) q^j \quad (q > 1). \tag{1.34}$$

Also (see [19], [25])

$$\int_0^{\infty/A} f(t)\, d_q t = (1-q) \sum_{k\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right). \tag{1.35}$$

It is noted from [20, p. 68, Theorem 19.1] and [20, p. 71, Proposition 19.1] that the improper q-integrals (1.33) and (1.34) converges if $t^{\alpha} f(t)$ is bounded in a neighborhood of $t = 0$ for some $0 \leq \alpha < 1$ and $t^{\beta} f(t)$ is bounded for sufficiently large t with some $\beta > 1$, respectively.

The formula for q-integration by parts is given as follows:

$$\int_a^b f(t) d_q g(t) = f(t)g(t)\Big|_a^b - \int_a^b g(qt) d_q f(t) \quad (0 \leq a < b \leq \infty). \tag{1.36}$$

Jackson [19] defined the q-gamma function $\Gamma_q(t)$ by

$$\Gamma_q(t) = \frac{(q;q)_\infty}{(q^t;q)_\infty} (1-q)^{1-t} = (1-q)_q^{t-1} (1-q)^{1-t} \quad (0 < q < 1;\, t > 0). \tag{1.37}$$

The correct integral representation of $\Gamma_q(t)$ (e.g., [25, Eq. (1.11)]) is

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} u^{t-1}\, \mathbf{E}_q(-qu)\, d_q u \quad (0 < q < 1,\, t > 0). \tag{1.38}$$

Another integral representation of q-gamma function denoted by ${}_q\Gamma(t)$ is given by (see [25, Eq. (1.19)])

$${}_q\Gamma(t) = K(A,t) \int_0^{\infty/A(1-q)} u^{t-1}\, \mathbf{e}_q(-u)\, d_q u \quad (0 < q < 1,\, t > 0), \tag{1.39}$$

where $K(\eta, t)$ is the notable function (see [25, Eq. (1.18)])

$$K(\eta,t) := \eta^{t-1} \frac{\left(-\frac{q}{\eta};q\right)_\infty (-\eta;q)_\infty}{\left(-\frac{q^t}{\eta};q\right)_\infty (-q^{1-t}\eta;q)_\infty} \quad (\eta \in \mathbb{C} \setminus \{0\},\, t \in \mathbb{C};\, 0 < q < 1). \tag{1.40}$$

The function $K(\eta,t)$ satisfies $K(q\eta,t)=K(\eta,t)$, which is meant to be a q-constant in η.

The series representations of $\Gamma_q(t)$ and ${}_q\Gamma(t)$ are given as follows:

$$\Gamma_q(t)=(q;q)_\infty(1-q)^{1-t}\sum_{k=0}^{\infty}\frac{q^{kt}}{(q;q)_k} \tag{1.41}$$

and

$${}_q\Gamma(t)=\frac{K(A;t)}{(1-q)^{t-1}\left(-\frac{1}{A};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(\frac{q^k}{A}\right)^t\left(-\frac{1}{A};q\right)_k, \tag{1.42}$$

where $0<q<1$ and $t>0$. Indeed, apply Euler's formula (see, e.g., [9, p. 490, Corollary 10.2.2]) to the first equality of (1.37) to give (1.41). Also, use the third equality of (1.11) and (1.35) in (1.39) to get (1.42).

Suppose that $f(t)$ is a real-(or complex-) valued function of the (time) variable $t>0$ and s is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

$$\begin{aligned}F(s)=\mathcal{L}\{f(t):s\}&=\int_0^\infty e^{-st}f(t)\,dt\\&=\lim_{\tau\to\infty}\int_0^\tau e^{-st}f(t)\,dt,\end{aligned} \tag{1.43}$$

whenever the limit exits (as a finite number). The so-called Sumudu transform is an integral transform which was defined and studied by Watugala [30] to facilitate the process of solving differential and integral equations in the time domain. The Sumudu transform has been used in various applications of system engineering and applied physics. For some fundamental properties of the Sumudu transform, one may refer to the works including (for example) [10-12, 30]. It turns out that the Sumudu transform has very special properties which are useful in solving problems involving kinetic equations in science and engineering. Let $\mathfrak{A}$ be the class of exponentially bounded functions $f:\mathbb{R}\to\mathbb{R}$, that is,

$$|f(t)|<\begin{cases}M\exp\left(-\dfrac{t}{\tau_1}\right) & (t\leqq 0),\\[2ex] M\exp\left(\dfrac{t}{\tau_2}\right) & (t\geqq 0),\end{cases} \tag{1.44}$$

where M, τ_1 and τ_2 are some positive real constants. The Sumudu transform defined on the set $\mathfrak{A}$ is given by the following formula (see [30]; see also [13], [22])

$$G(u) = \mathcal{S}\left[f(t); u\right] := \int_0^\infty e^{-t}\, f(ut)\, dt \qquad (-\tau_1 < u < \tau_2). \tag{1.45}$$

The Sumudu transform given in (1.45) can also be derived directly from the Fourier integral. Moreover, it can be easily verified that the Sumudu transform is a linear operator and the function $G(u)$ in (1.45) keeps the same units as $f(t)$; that is, for any real or complex number λ, we have

$$\mathcal{S}[f(\lambda t); u] = G(\lambda u).$$

The Sumudu transform $G(u)$ and the Laplace transform $F(s)$ exhibit a duality relation that may be expressed as follows:

$$G\left(\frac{1}{s}\right) = s\, F(s) \qquad \text{or} \qquad G(u) = \frac{1}{u}\, F\left(\frac{1}{u}\right). \tag{1.46}$$

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform.

The q-analogues of the Laplace transform (1.43) of a function $f(t)$ are defined by (see, e.g., [6], [8], [15], [24])

$$L_q\{f(t); s\} = \frac{1}{1-q} \int_0^{\frac{1}{s}} \mathbf{E}_q\left(-\frac{qst}{1-q}\right) f(t)\, d_q t \tag{1.47}$$

and

$${}_qL\{f(t); s\} = \frac{1}{1-q} \int_0^{\infty} \mathbf{e}_q\left(-\frac{st}{1-q}\right) f(t)\, d_q t. \tag{1.48}$$

Use (1.30) and (1.33) in (1.47) and (1.48) with (1.6), respectively, to give the following summation representations:

$$L_q\{f(t); s\} = \frac{(q;q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q;q)_j}\, f\left(q^j/s\right) \tag{1.49}$$

and

$${}_qL\{f(t); s\} = \frac{1}{(-s;q)_\infty} \sum_{j\in\mathbb{Z}} (-s;q)_j\, q^j\, f\left(q^j\right). \tag{1.50}$$

The bilateral summation (1.50) may be a corrected form of the corresponding formula in [2, p. 243].

Khan and Khan [21] introduced and investigated N-transform of $f(t)$ defined on $[0,\infty)$, which is a new and interesting integral transform which may combine the Laplace and Sumudu transforms, defined by

$$\mathsf{N}\left\{f(t)\right\}(u;v)=\int_0^\infty e^{-vt}\,f\left(ut\right)dt \tag{1.51}$$

provided this integral converges. Obviously, for u, $v>0$,

$$\mathsf{N}\left\{f(t)\right\}(u;v)=\frac{1}{u}\int_0^\infty \exp\left(-\frac{vt}{u}\right)f(t)\,dt=\frac{1}{u}\mathcal{L}\left\{f\left(t\right):\frac{v}{u}\right\}, \tag{1.52}$$

and

$$\mathsf{N}\left\{f(t)\right\}(u;v)=\frac{1}{v}\int_0^\infty e^{-t}f\left(\frac{ut}{v}\right)\,dt=\frac{1}{v}\mathcal{S}\left[f\left(t\right);\frac{u}{v}\right]\quad(f(t)\in\mathfrak{A}). \tag{1.53}$$

Let $\mathfrak{A}_1$ be the class of $\mathbf{E}_q$-bounded functions $f:\mathbb{R}\to\mathbb{R}$, that is,

$$|f(t)|<\begin{cases} M\,\mathbf{E}_q\left(\dfrac{t}{(1-q)\tau_1}\right) & (t\leqq 0),\\[2ex] M\,\mathbf{E}_q\left(-\dfrac{t}{(1-q)\tau_2}\right) & (t\geqq 0),\end{cases} \tag{1.54}$$

where M, τ_1 and τ_2 are some positive real constants. Albayrak et al. [1, 2] introduced the q-analogue of Sumudu transform of the function $f(t)\in\mathfrak{A}_1$ as follows (see also [7], [28]):

$$S_q\left\{f(t);s\right\}=\frac{1}{(1-q)s}\int_0^s \mathbf{E}_q\left(-\frac{qt}{(1-q)s}\right)f(t)\,d_qt\quad\left(s\in(-\tau_1,\tau_2)\right). \tag{1.55}$$

Let $\mathfrak{A}_2$ be the class of $\mathbf{e}_q$-bounded functions $f:\mathbb{R}\to\mathbb{R}$, that is,

$$|f(t)|<\begin{cases} M\,\mathbf{e}_q\left(-\dfrac{t}{(1-q)\tau_1}\right) & (t\leqq 0),\\[2ex] M\,\mathbf{e}_q\left(\dfrac{t}{(1-q)\tau_2}\right) & (t\geqq 0),\end{cases} \tag{1.56}$$

where M, τ_1 and τ_2 are some positive real constants. Albayrak et al. [1, 2] introduced another q-analogue of Sumudu transform of the function $f(t)\in\mathfrak{A}_2$ as follows:

$${}_qS\left\{f(t);s\right\}=\frac{1}{(1-q)s}\int_0^\infty \mathbf{e}_q\left(-\frac{t}{(1-q)s}\right)f(t)\,d_qt\quad\left(s\in(-\tau_1,\tau_2)\right). \tag{1.57}$$

Al-Omari [5] introduced q-analogues of the N-transform as follows:

$$\mathsf{N}_q\left\{f(t)\right\}(u;v)=\frac{1}{(1-q)u}\int_0^{\frac{u}{v}} f(t)\,\mathbf{E}_q\left(-\frac{qv}{(1-q)u}t\right)d_qt \quad (f\in\mathfrak{A}_1) \tag{1.58}$$

and

$$_q\mathsf{N}\left\{f(t)\right\}(u;v)=\frac{1}{1-q}\int_0^{\infty} f(t)\,\mathbf{e}_q\left(-\frac{v}{(1-q)u}t\right)d_qt \quad (f\in\mathfrak{A}_2)\,. \tag{1.59}$$

Remark 1.2. *Use (1.5), (1.12), (1.30), and (1.58) to give*

$$\mathsf{N}_q\left\{f(t)\right\}(u;v)=\frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\,f\left(q^k\frac{u}{v}\right) \quad (f\in\mathfrak{A}_1)\,. \tag{1.60}$$

Similarly, we have

$$_q\mathsf{N}\{f(t)\}(u;v)=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(-\frac{v}{u};q\right)_k q^k\,f\left(q^k\right) \quad (f\in\mathfrak{A}_2)\,. \tag{1.61}$$

The Humbert function $J_{m,n}(x)$ is defined by means of the generating function (see, e.g., [17, 18, 23, 27, 29])

$$\exp\left[\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right]=\sum_{m,\,n=-\infty}^{\infty}J_{m,n}(x)\,u^m\,t^n. \tag{1.62}$$

Or, explicitly,

$$\begin{aligned}J_{m,n}(x)&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(m+k+1)\Gamma(n+k+1)}\left(\frac{x}{3}\right)^{m+n+3k}\\&=\left(\frac{x}{3}\right)^{m+n}\frac{1}{\Gamma(m+1)\Gamma(n+1)}\,{}_0F_2\left(-;\,m+1,\,n+1\,;\,-\frac{x^3}{27}\right).\end{aligned} \tag{1.63}$$

Srivastava and Shehata [27] introduced the following three q-extensions of the Humbert functions: The q-Humbert function $\mathscr{H}^{(1)}_{m,n}(x|q)$ of the first kind is defined by means of the generating function

$$\mathbf{e}_q\left(\frac{xu}{3}\right)\mathbf{e}_q\left(\frac{xt}{3}\right)\mathbf{e}_q\left(-\frac{x}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}\mathscr{H}^{(1)}_{m,n}(x|q)\,u^m\,t^n. \tag{1.64}$$

An explicit representation of $\mathscr{H}^{(1)}_{m,n}(x|q)$ is given by

$$\mathscr{H}^{(1)}_{m,n}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k \ (q;q)_{m+k} \ (q;q)_{n+k}} \left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.65}$$

The q-Humbert function $\mathscr{H}^{(2)}_{m,n}(x|q)$ of the second kind is defined by means of the generating function

$$\mathbf{E}_q\left(\frac{xu}{3}\right)\mathbf{E}_q\left(\frac{xt}{3}\right)\mathbf{E}_q\left(-\frac{qx}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} q^{\binom{m}{2}+\binom{n}{2}}\mathscr{H}^{(2)}_{m,n}(x|q)\, u^m\, t^n. \tag{1.66}$$

An explicit representation of $\mathscr{H}^{(2)}_{m,n}(x|q)$ is given by

$$\mathscr{H}^{(2)}_{m,n}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k \ (q;q)_{m+k} \ (q;q)_{n+k}} q^{\frac{1}{2}k[3k+2(m+n)-1]} \left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.67}$$

The q-Humbert function $\mathscr{H}^{(3)}_{m,n}(x|q)$ of the third kind is defined by means of the generating function

$$\mathbf{e}_q\left(\frac{xu}{3}\right)\mathbf{e}_q\left(\frac{xt}{3}\right)\mathbf{E}_q\left(-\frac{qx}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} \mathscr{H}^{(3)}_{m,n}(x|q)\, u^m\, t^n. \tag{1.68}$$

An explicit representation of $\mathscr{H}^{(3)}_{m,n}(x|q)$ is given by

$$\mathscr{H}^{(3)}_{m,n}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k \ (q;q)_{m+k} \ (q;q)_{n+k}} q^{k+1} \left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.69}$$

Remark 1.3. *Using the ratio test to the series expressions of the q-Humbert functions of the first, second, and third kinds (1.65), (1.67), and (1.69), we find that, for* $0 < q < 1$*,*

(i) The series $\mathscr{H}^{(1)}_{m,n}(x|q)$ *in (1.65) converges absolutely for* $|x| < \frac{3}{1-q}$*;*

(ii) The series $\mathscr{H}^{(2)}_{m,n}(x|q)$ *in (1.67) converges absolutely for* $|x| < \infty$*;*

(iii) The series $\mathscr{H}^{(3)}_{m,n}(x|q)$ *in (1.69) converges absolutely for* $|x| < \frac{3}{\sqrt[3]{q}\,(1-q)}$*.*

For q-analogues of other transforms, one may be referred (for example) to [3], [4].

In this paper, we aim to provide q-natural transforms of a finite product of the q-Humbert functions. Among numerous particular cases of our main identities, some of them are also considered.

2. $\mathbf{N}_q$-Transforms of the q-Humbert functions

The $\mathbf{N}_q$-transforms of the q-Humbert functions are given. Here and elsewhere, a multiple-valued function assumes to be taken as its principal value.

Theorem 2.1. *Let $0 < q < 1$. Also let $n \in \mathbb{N}$, u, $v \in \mathbb{R}^+$, and $a_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, n)$, $\omega \in \mathbb{C}$ with $\Re(\omega) > 1$. Further let μ_j, $\nu_j \in \mathbb{Z}$ with $\Re(\omega + \mu_j + \nu_j) > 0$ $(j = 1, \ldots, n)$, and $\frac{u}{v} < \frac{1}{\mathbf{a}(1-q)^3}$ where $\mathbf{a} = \min\{|a_j| : j = 1, \ldots, n\}$. Then the $\mathbf{N}_q$-transform of the q-Humbert functions of the first kind is given by*

$$\begin{aligned}\mathbf{N}_q\left\{t^{\omega-1}\prod_{j=1}^{n}\mathscr{H}^{(1)}_{3\mu_j,3\nu_j}\left(3(a_jt)^{\frac{1}{3}}\,\middle|\,q\right)\right\}(u;v) &= \frac{1}{v}\left(\frac{(1-q)u}{v}\right)^{w-1}\\ &\times\prod_{j=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,\left((1-q)a_j\frac{u}{v}\right)^{\mu_j+\nu_j+\ell}}{[\ell]_q!\,[3\mu_j+\ell]_q!\,[3\nu_j+\ell]_q!}\,\Gamma_q\left(\omega+\mu_j+\nu_j+\ell\right).\end{aligned} \tag{2.1}$$

Proof. Let

$$f_1(t) := t^{\omega-1}\prod_{j=1}^{n}\mathscr{H}^{(1)}_{3\mu_j,3\nu_j}\left(3(a_jt)^{\frac{1}{3}}\,\middle|\,q\right). \tag{2.2}$$

Then employ (1.65) in $f_1(t)$ in (2.2) to get

$$f_1(t) = \sum_{k=0}^{\infty}\prod_{j=1}^{n}\frac{(-1)^k\,(1-q)^{3\mu_j+3\nu_j+3k}\,a_j^{\mu_j+\nu_j+k}\,t^{\mu_j+\nu_j+k+\omega-1}}{(q;q)_k\,(q;q)_{3\mu_j+k}\,(q;q)_{3\nu_j+k}}. \tag{2.3}$$

Since $0 < t < \frac{u}{v}$, we have $|t| < \frac{1}{\mathbf{a}(1-q)^3}$. we find from **Remark** 1.3, (i) and assumptions that $f_1 \in \mathfrak{A}_1$. Then using $f_1(t)$ in (2.3) to Eq. (1.60) and simplifying the resulting identity with the aid of some chosen identities given in Section 1, we can obtain the desired identity (2.1).

Theorem 2.2. *Let $0 < q < 1$. Also let $n \in \mathbb{N}$, u, $v \in \mathbb{R}^+$, and $a_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, n)$, $\omega \in \mathbb{C}$ with $\Re(\omega) > 1$. Further let μ_j, $\nu_j \in \mathbb{Z}$ with $\Re(\omega + \mu_j + \nu_j) > 0$ $(j = 1, \ldots, n)$. Then the $\mathbf{N}_q$-transform of the q-Humbert functions of the second kind is given by*

$$\begin{aligned}\mathbf{N}_q\left\{t^{\omega-1}\prod_{j=1}^{n}\mathscr{H}^{(2)}_{3\mu_j,3\nu_j}\left(3(a_jt)^{\frac{1}{3}}\,\middle|\,q\right)\right\}(u;v) &= \frac{1}{v}\left(\frac{(1-q)u}{v}\right)^{w-1}\\ &\times\prod_{j=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,q^{\frac{\ell}{2}[3\ell+2(3\mu_j+3\nu_j)-1]}\,\left((1-q)a_j\frac{u}{v}\right)^{\mu_j+\nu_j+\ell}}{[\ell]_q!\,[3\mu_j+\ell]_q!\,[3\nu_j+\ell]_q!}\,\Gamma_q\left(\omega+\mu_j+\nu_j+\ell\right).\end{aligned} \tag{2.4}$$

Proof. Using (1.60) and (1.67), similarly as in the proof of Theorem 2.1, we can prove the identity (2.4). We omit the details.

Theorem 2.3. *Let* $0 < q < 1$. *Also let* $n \in \mathbb{N}$, $u, v \in \mathbb{R}^+$, *and* $a_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, n)$, $\omega \in \mathbb{C}$ *with* $\Re(\omega) > 1$. *Further let* $\mu_j, \nu_j \in \mathbb{Z}$ *with* $\Re(\omega + \mu_j + \nu_j) > 0$ $(j = 1, \ldots, n)$, *and* $\frac{u}{v} < \frac{q}{\mathbf{a}(1-q)^3}$ *where* $\mathbf{a} = \min\{|a_j| : j = 1, \ldots, n\}$. *Then the* $\mathbf{N}_q$*-transform of the q-Humbert functions of the third kind is given by*

$$\begin{aligned}\mathbf{N}_q\left\{t^{\omega-1}\prod_{j=1}^{n} q^{\mu_j+\nu_j}\mathscr{H}^{(3)}_{3\mu_j,3\nu_j}\left(3\left(q^{-2}a_j t\right)^{\frac{1}{3}} \mid q\right)\right\}(u;v) &= \frac{q}{v}\left(\frac{(1-q)u}{v}\right)^{w-1} \\ \times \prod_{j=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left(\frac{(1-q)u}{qv}a_j\right)^{\mu_j+\nu_j+\ell}}{[\ell]_q!\,[3\mu_j+\ell]_q!\,[3\nu_j+\ell]_q!}\,\Gamma_q\left(\omega+\mu_j+\nu_j+\ell\right).\end{aligned} \tag{2.5}$$

Proof. Use (1.60) and (1.69). The proof would run parallel with that of Theorem 2.1. The details are omitted.

3. ${}_q\mathbf{N}$-Transforms of the q-Humbert functions

The ${}_q\mathbf{N}$-transforms of the q-Humbert functions are presented.

Theorem 3.1. *Let* $0 < q < 1$. *Also let* $n \in \mathbb{N}$, $u, v \in \mathbb{R}^+$, *and* $a_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, n)$, $\omega \in \mathbb{C}$ *with* $\Re(\omega) > 1$. *Further let* $\mu_j, \nu_j \in \mathbb{Z}$ *with* $\Re(\omega + \mu_j + \nu_j) > 0$ $(j = 1, \ldots, n)$, *and* $\frac{u}{v} < \frac{1}{\mathbf{a}(1-q)^3}$ *where* $\mathbf{a} = \min\{|a_j| : j = 1, \ldots, n\}$. *Then the* ${}_q\mathbf{N}$*-transform of the q-Humbert functions of the first kind is given by*

$$\begin{aligned}{}_q\mathbf{N}\left\{t^{\omega-1}\prod_{j=1}^{n}\mathscr{H}^{(1)}_{3\mu_j,3\nu_j}\left(3(a_j t)^{\frac{1}{3}} \mid q\right)\right\}(u;v) &= \frac{1}{1-q}\left(\frac{(1-q)u}{v}\right)^{w} \\ \times \prod_{j=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left((1-q)a_j\frac{u}{v}\right)^{\mu_j+\nu_j+\ell}}{[\ell]_q!\,[3\mu_j+\ell]_q!\,[3\nu_j+\ell]_q!}\,\frac{{}_q\Gamma\left(\omega+\mu_j+\nu_j+\ell\right)}{K\left(\frac{u}{v};\omega+\mu_j+\nu_j+\ell\right)}.\end{aligned} \tag{3.1}$$

Proof. Use (1.42), (1.61), and (1.65). We can prove the identity (3.1) as in the proof of Theorem 2.1. We omit the details.

Theorem 3.2. *Let* $0 < q < 1$. *Also let* $n \in \mathbb{N}$, $u, v \in \mathbb{R}^+$, *and* $a_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, n)$, $\omega \in \mathbb{C}$ *with* $\Re(\omega) > 1$. *Further let* $\mu_j, \nu_j \in \mathbb{Z}$ *with* $\Re(\omega + \mu_j + \nu_j) > 0$ $(j = 1, \ldots, n)$, *and* $\frac{u}{v} < \frac{q}{\mathbf{a}(1-q)^3}$ *where* $\mathbf{a} = \min\{|a_j| : j = 1, \ldots, n\}$. *Then the*

${}_q\mathbf{N}$-transform of the q-Humbert functions of the third kind is given by

$$\begin{aligned}
&{}_q\mathbf{N}\left\{t^{\omega-1}\prod_{j=1}^{n} q^{\mu_j+\nu_j}\mathscr{H}^{(3)}_{3\mu_j,3\nu_j}\left(3\left(q^{-2}a_j t\right)^{\frac{1}{3}}\mid q\right)\right\}(u;v)=\frac{q}{1-q}\left(\frac{(1-q)u}{v}\right)^{w-1}\\
&\times\prod_{j=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left(\frac{(1-q)u}{qv}a_j\right)^{\mu_j+\nu_j+\ell}}{[\ell]_q!\,[3\mu_j+\ell]_q!\,[3\nu_j+\ell]_q!}\,\frac{{}_q\Gamma\left(\omega+\mu_j+\nu_j+\ell\right)}{K\left(\frac{u}{v};\omega+\mu_j+\nu_j+\ell\right)}.
\end{aligned}\tag{3.2}$$

Proof. Use (1.42), (1.61), and (1.69). The proof would run parallel with that of Theorem 2.1. The details are omitted.

4. Special cases

Among numerous particular cases of the main identities in Sections 2 and 3, we consider only the case $n=1$ with $a_1=a$, $\mu_1=\mu$, and $\nu_1=\nu$ in the following corollaries.

Corollary 4.1. *Let $0<q<1$. Also let $u,\, v\in\mathbb{R}^+$, $a\in\mathbb{C}\setminus\{0\}$, and $\omega\in\mathbb{C}$ with $\Re(\omega)>1$. Further let $\mu,\,\nu\in\mathbb{Z}$ with $\Re\left(\omega+\mu+\nu\right)>0$, and $\frac{u}{v}<\frac{1}{\mathbf{a}(1-q)^3}$ where $\mathbf{a}=|a|$. Then the $\mathbf{N}_q$-transform of the q-Humbert functions of the first kind is given by*

$$\begin{aligned}
&\mathbf{N}_q\left\{t^{\omega-1}\mathscr{H}^{(1)}_{3\mu,3\nu}\left(3(at)^{\frac{1}{3}}\mid q\right)\right\}(u;v)=\frac{1}{v}\left(\frac{(1-q)u}{v}\right)^{w-1}\\
&\times\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left((1-q)a\frac{u}{v}\right)^{\mu+\nu+\ell}}{[\ell]_q!\,[3\mu+\ell]_q!\,[3\nu+\ell]_q!}\,\Gamma_q\left(\omega+\mu+\nu+\ell\right).
\end{aligned}\tag{4.1}$$

Corollary 4.2. *Let $0<q<1$. Also let $u,\, v\in\mathbb{R}^+$, $a\in\mathbb{C}\setminus\{0\}$, and $\omega\in\mathbb{C}$ with $\Re(\omega)>1$. Further let $\mu,\,\nu\in\mathbb{Z}$ with $\Re\left(\omega+\mu+\nu\right)>0$. Then the $\mathbf{N}_q$-transform of the q-Humbert functions of the second kind is given by*

$$\begin{aligned}
&\mathbf{N}_q\left\{t^{\omega-1}\mathscr{H}^{(2)}_{3\mu,3\nu}\left(3(at)^{\frac{1}{3}}\mid q\right)\right\}(u;v)=\frac{1}{v}\left(\frac{(1-q)u}{v}\right)^{w-1}\\
&\times\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,q^{\frac{\ell}{2}[3\ell+2(3\mu+3\nu)-1]}\left((1-q)a\frac{u}{v}\right)^{\mu+\nu+\ell}}{[\ell]_q!\,[3\mu+\ell]_q!\,[3\nu+\ell]_q!}\,\Gamma_q\left(\omega+\mu+\nu+\ell\right).
\end{aligned}\tag{4.2}$$

Corollary 4.3. *Let $0<q<1$. Also let $u,\, v\in\mathbb{R}^+$, $a\in\mathbb{C}\setminus\{0\}$, and $\omega\in\mathbb{C}$ with $\Re(\omega)>1$. Further let $\mu,\,\nu\in\mathbb{Z}$ with $\Re\left(\omega+\mu+\nu\right)>0$, and $\frac{u}{v}<\frac{q}{\mathbf{a}(1-q)^3}$ where*

$\mathbf{a} = |a|$. *Then the* $\mathbf{N}_q$*-transform of the* q*-Humbert functions of the third kind is given by*

$$\begin{aligned}\mathbf{N}_q\left\{t^{\omega-1}\mathscr{H}^{(3)}_{3\mu,3\nu}\left(3\left(q^{-2}at\right)^{\frac{1}{3}} \,\middle|\, q\right)\right\}(u;v) &= \frac{q}{v}\left(\frac{(1-q)u}{v}\right)^{w-1} \\ &\times\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left(\frac{(1-q)u}{qv}a\right)^{\mu+\nu+\ell}}{[\ell]_q!\,[3\mu+\ell]_q!\,[3\nu+\ell]_q!}\,\Gamma_q\left(\omega+\mu+\nu+\ell\right).\end{aligned} \tag{4.3}$$

Corollary 4.4. *Let* $0 < q < 1$*. Also let* $u, v \in \mathbb{R}^+$*,* $a \in \mathbb{C}\setminus\{0\}$*, and* $\omega \in \mathbb{C}$ *with* $\Re(\omega) > 1$*. Further let* $\mu, \nu \in \mathbb{Z}$ *with* $\Re(\omega+\mu+\nu) > 0$*, and* $\frac{u}{v} < \frac{1}{\mathbf{a}(1-q)^3}$ *where* $\mathbf{a} = |a|$*. Then the* ${}_q\mathbf{N}$*-transform of the* q*-Humbert functions of the first kind is given by*

$$\begin{aligned}{}_q\mathbf{N}\left\{t^{\omega-1}\mathscr{H}^{(1)}_{3\mu,3\nu}\left(3(at)^{\frac{1}{3}} \,\middle|\, q\right)\right\}(u;v) &= \frac{1}{1-q}\left(\frac{(1-q)u}{v}\right)^{w} \\ &\times\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left((1-q)a\frac{u}{v}\right)^{\mu+\nu+\ell}}{[\ell]_q!\,[3\mu+\ell]_q!\,[3\nu+\ell]_q!}\,\frac{{}_q\Gamma\left(\omega+\mu+\nu+\ell\right)}{K\left(\frac{u}{v};\omega+\mu+\nu+\ell\right)}.\end{aligned} \tag{4.4}$$

Corollary 4.5. *Let* $0 < q < 1$*. Also let* $u, v \in \mathbb{R}^+$*,* $a \in \mathbb{C}\setminus\{0\}$*, and* $\omega \in \mathbb{C}$ *with* $\Re(\omega) > 1$*. Further let* $\mu, \nu \in \mathbb{Z}$ *with* $\Re(\omega+\mu+\nu) > 0$*, and* $\frac{u}{v} < \frac{q}{\mathbf{a}(1-q)^3}$ *where* $\mathbf{a} = |a|$*. Then the* ${}_q\mathbf{N}$*-transform of the* q*-Humbert functions of the third kind is given by*

$$\begin{aligned}{}_q\mathbf{N}\left\{t^{\omega-1}q^{\mu+\nu}\mathscr{H}^{(3)}_{3\mu,3\nu}\left(3\left(q^{-2}at\right)^{\frac{1}{3}} \,\middle|\, q\right)\right\}(u;v) &= \frac{q}{1-q}\left(\frac{(1-q)u}{v}\right)^{w-1} \\ &\times\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\left(\frac{(1-q)u}{qv}a\right)^{\mu+\nu+\ell}}{[\ell]_q!\,[3\mu+\ell]_q!\,[3\nu+\ell]_q!}\,\frac{{}_q\Gamma\left(\omega+\mu+\nu+\ell\right)}{K\left(\frac{u}{v};\omega+\mu+\nu+\ell\right)}.\end{aligned} \tag{4.5}$$

5. Concluding remarks

The q-natural transforms of a finite product of the q-Humbert functions are investigated. The main identities in this paper are easily recognized to yield diverse and numerous special identities. For example, the cases $u = 1$ and $v = 1$ of the main results in Sections 2 and 3 can yield, respectively, the corresponding q-Laplace and q-Sumudu transform formulas.

Posing a problem. Establish the ${}_q\mathbf{N}$-transform of the q-Humbert functions of the second kind as in Section 3.

De Sole and Kac [25, p. 13] commented that the simplest proof of the second equalities in (1.11) and (1.12) due to Euler can be conducted by using the q-analogue of Taylor's formula (see, e.g., [26, p. 486, Theorem 6.3], [25, p. 75, Theorem 20.2]). Here we try to prove (1.12) by using the q-analogue of Taylor's formula. Indeed, let

$$f(t) = (-(1-q)t;q)_\infty = \prod_{k=0}^{\infty} \left(1 + t(1-q)q^k\right) \quad (0 < q < 1).$$

Applying $f(t)$ to the q-analogue of Taylor's formula, we obtain

$$f(t) = \sum_{j=0}^{n} q^{\binom{j}{2}} \frac{b^j}{[j]_q!} + R_{n,q}, \tag{5.1}$$

where

$$R_{n,q} = \frac{1}{[n]_q!} \int_0^b f\left(q^{n+1}t\right) (b - qt)_q^n \, d_q t \quad \left(b \in \mathbb{R}^+\right).$$

Using (1.23) and (1.30), we have

$$R_{n,q} = \frac{(1-q)\, b^{n+1}}{[n]_q!} \sum_{k=0}^{\infty} q^k f\left(q^{n+1+k} b\right) \prod_{\mu=0}^{n-1} \left(1 - q^{k+\mu+1}\right). \tag{5.2}$$

We find that

$$\prod_{\mu=0}^{n-1} \left(1 - q^{k+\mu+1}\right) \le \prod_{\mu=0}^{n-1} \left(1 - q^{k+n-1+1}\right) = \left(1 - q^{n+k}\right)^n$$

and

$$f\left(q^{n+1+k} b\right) = \prod_{j=0}^{\infty} \left(1 + (1-q) q^{n+1+k} q^j b\right) \le \prod_{j=0}^{\infty} \left(1 + b(1-q)q^j\right) = f(b)$$

Apply these two inequalities in (5.2) to give

$$R_{n,q} \le (1-q) f(b) \frac{b^{n+1}}{[n]_q!} \sum_{k=0}^{\infty} q^k \left(1 - q^{n+k}\right)^n. \tag{5.3}$$

We observe that

$$\begin{aligned}\sum_{k=0}^{\infty} q^k \left(1-q^{n+k}\right)^n &= \sum_{k=0}^{\infty} q^k \sum_{j=0}^{n}(-1)^j \binom{n}{j} q^{j(n+k)} \\ &= \sum_{j=0}^{n}(-1)^j \binom{n}{j} q^{jn} \sum_{k=0}^{\infty} q^{k(j+1)} \\ &= \sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{q^{jn}}{1-q^{j+1}},\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{q^{jn}}{1-q^{j+1}}\right| &\leq \sum_{j=0}^{n} \binom{n}{j} \frac{q^{jn}}{1-q^{j+1}} \leq \sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{q^{jn}}{1-q} \\ = O\left((1+q^n)^n\right) = O\left(q^{n^2}\right) &\leq O\left(q^{n(n-1)}\right) \leq O\left(q^{\frac{n(n-1)}{2}}\right)\end{aligned} \tag{5.4}$$

for sufficiently large $n \in \mathbb{N}$. Employing the inequality (5.4) in (5.3), we get

$$R_{n,q} = O\left(q^{\binom{n}{2}} \frac{b^n}{[n]_q!}\right) \tag{5.5}$$

for sufficiently large $n \in \mathbb{N}$. Since the radius of convergence of the series in the first equality in (1.12) is ∞, we find from (5.5) that $R_{n,q} \to 0$ as $n \to \infty$ for all $b \in \mathbb{R}^+$. In view of (5.1) and (1.12), we get

$$\mathbf{E}_q(b) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{b^n}{[n]_q!} = (-(1-q)b;q)_\infty \quad \left(b \in \mathbb{R}^+\right). \tag{5.6}$$

Finally, by the principle of analytic continuation, the second equality in (5.6) holds true for all $z \in \mathbb{C}$.

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